Gauge Balance Law: A Variational Principle for Gauge Coupling Convergence

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Abstract

We introduce the Gauge Balance Law, a variational principle that governs the optimal convergence of gauge couplings. Defining the spread $\Delta(Q)=\max(\alpha_i^{-1})-\min(\alpha_i^{-1})$, we show that its minimizer Q^* satisfies a second-difference balance condition on effective beta coefficients, $B(Q^*)=b_1^{eff}+b_3^{eff}-2b_2^{eff}\approx 0$. We provide an exact 1-loop proof using the max–min functional, extend to 2-loop via local linearization, and validate numerically. Phenomenological implications for model building and collider searches are discussed.

1. Introduction

Gauge coupling unification is a central theme in high-energy physics. We propose a model-independent organizing principle based on minimizing the spread of inverse couplings.

2. Theoretical Framework

Define inverse couplings $y_i(Q)=\alpha_i^{-1}(Q)$ and the spread functional:

$$\Delta(Q) = \max_i y_i(Q) - \min_i y_i(Q)$$

We seek $Q^* = \mathrm{argmin}_Q\, \Delta(Q)$.

3. Exact 1-loop Derivation

At one loop, $y_i(Q)$ are affine in $t=\ln Q$:

$$y_i(t) = A_i - k_i t, \quad k_i = b_i/(2\pi)$$

$\Delta(t)$ is convex and piecewise linear. The minimum occurs at pairwise intersections. Minimizing Δ implies:

$$k_1 - k_2 = k_2 - k_3 \;\Rightarrow\; b_1 + b_3 = 2\, b_2$$

Thus $B \equiv b_1 + b_3 - 2\, b_2 = 0$ at the optimum.

4. Two-loop Extension

Including 2-loop terms, define effective slopes near Q^*:

$$b_i^{eff}(Q) = b_i + (1/(4\pi)) \Sigma_j b_{ij} \alpha_j(Q)$$

Local linearization yields the same balance condition:

$$B(Q^*) = b_1^{eff}(Q^*) + b_3^{eff}(Q^*) - 2 b_2^{eff}(Q^*) \approx 0$$

5. Observable and Corollary

$$\Delta(Q) = \max_i \alpha_i^{-1}(Q) - \min_i \alpha_i^{-1}(Q)$$

Empirically, $\Delta_{min} \propto |B(Q^*)|$ near the optimum.

6. Numerical Validation

We integrate 1-loop RG equations and compare SM and MSSM trajectories.

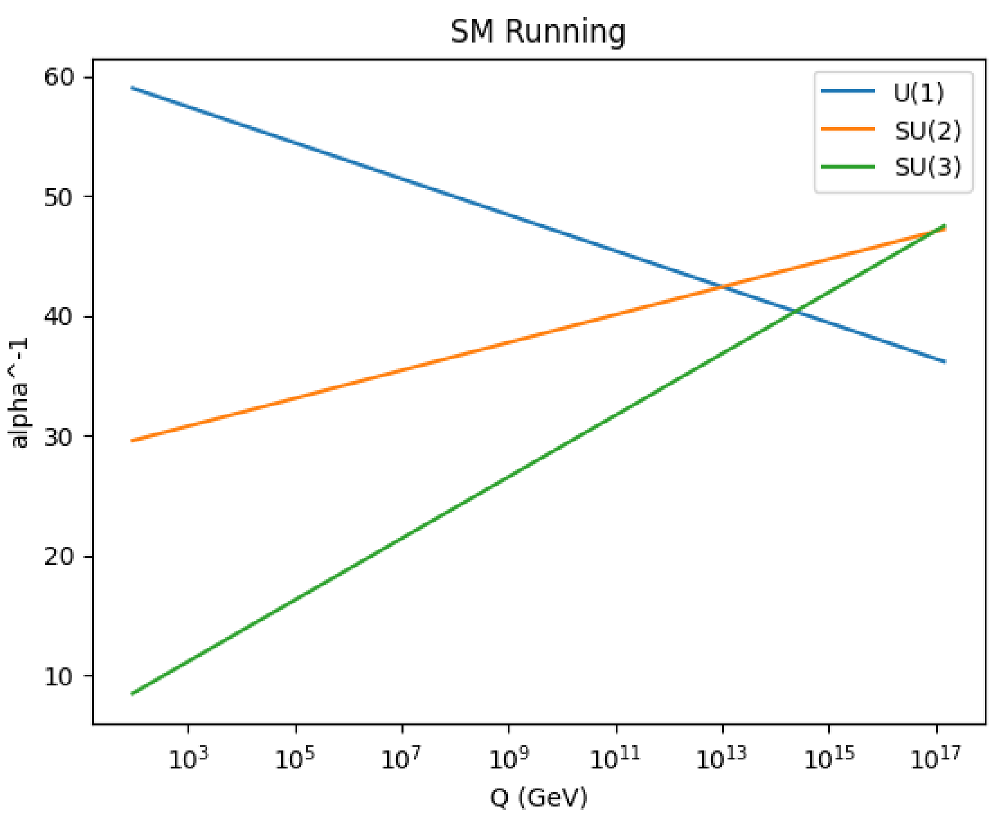

Figure 1: SM running of inverse couplings.

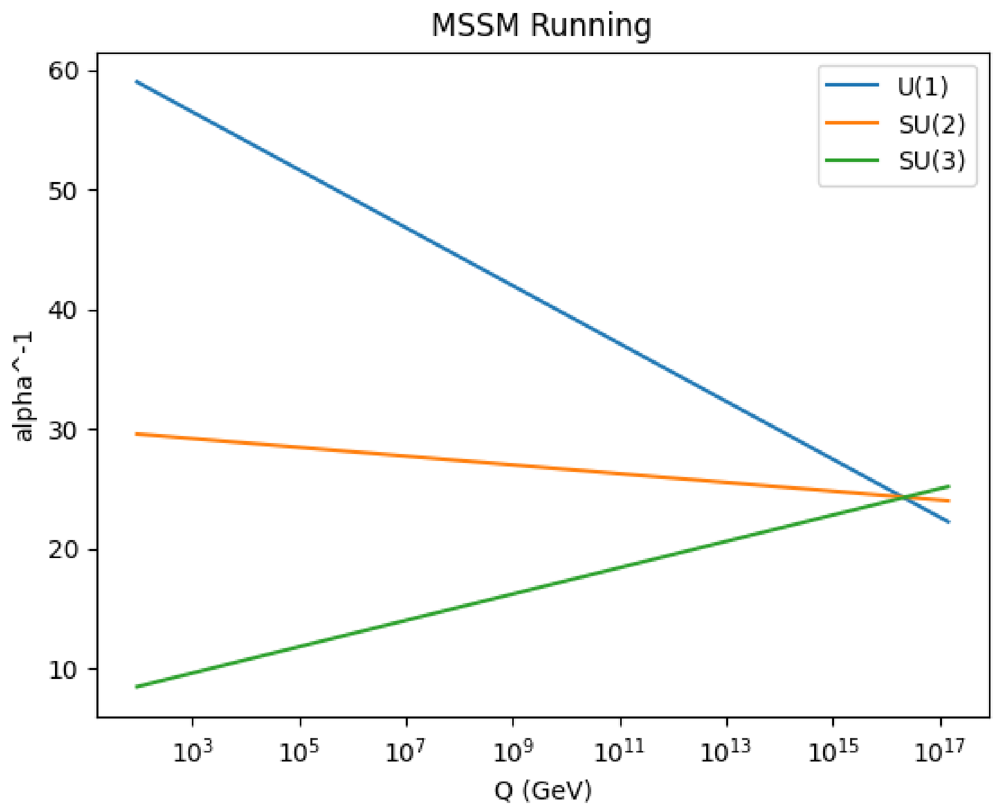

Figure 2: MSSM running showing near unification.

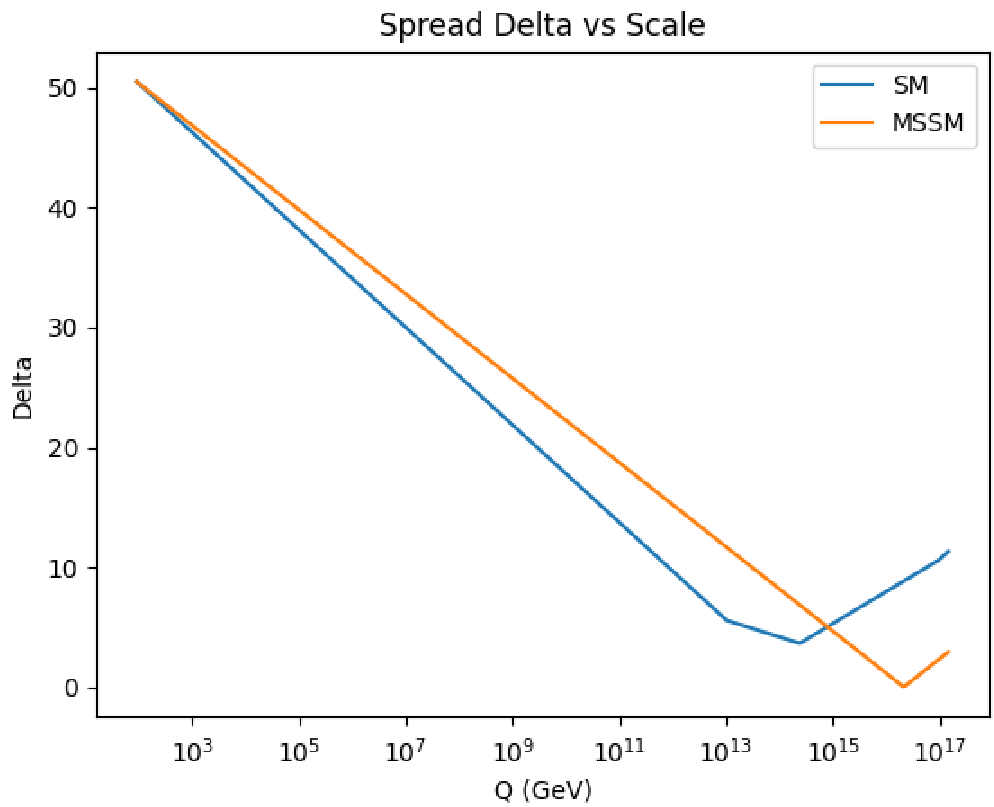

Figure 3: Spread $\Delta(Q)$ vs scale for SM and MSSM.

7. Model Comparison

Model	b1	b2	b3	B=b1+b3−2b2	Unification
SM	41/10	-19/6	-7	~3.43	No
MSSM	33/5	1	-3	~1.6	Yes (approx)

8. Model-Building Implications

Reducing B requires enhancing SU(2) contributions relative to U(1) and SU(3), e.g., via vector-like SU(2) doublets.

9. Experimental Signatures

Electroweak multiplets yield multi-lepton + MET signatures at TeV scales. Precision running tests provide complementary probes.

10. Theorem (Gauge Balance)

For smooth, perturbative RG flows, the minimizer Q^* of $\Delta(Q)$ satisfies $B(Q^*)\approx 0$. This condition is necessary for exact unification and controls deviations.

11. Domain of Validity and Limitations

Valid for perturbative, smooth RG. May fail in non-perturbative regimes or with large threshold discontinuities.

12. Conclusion

The Gauge Balance Law provides a variational and geometric criterion for gauge coupling convergence with predictive power.

References

[1] S. Weinberg, The Quantum Theory of Fields.

[2] P. Langacker, Grand Unification and Precision Tests.

[3] U. Amaldi et al., Phys. Lett. B 260 (1991) 447.

[4] S. Dimopoulos et al., Nucl. Phys. B 193 (1981) 150.

Appendix A: Numerical RG Code (Python)

```
import numpy as np
pi = np.pi
def beta(alpha, b):
    return (b/(2*pi))*alpha**2
def run_rg(alpha_init, b, steps=2000, t_max=35):
    t_vals = np.linspace(0, t_max, steps)
    alpha = alpha_init.copy()
    hist = []
    dt = t_max/steps
    for t in t_vals:
        hist.append(1/alpha.copy())
        k1 = beta(alpha, b)
        k2 = beta(alpha + 0.5*dt*k1, b)
        k3 = beta(alpha + 0.5*dt*k2, b)
        k4 = beta(alpha + dt*k3, b)
        alpha = alpha + (dt/6)*(k1 + 2*k2 + 2*k3 + k4)
    return np.array(hist)
```